Laplace Transformation

Basic formulae

1) $(a+b)^2 = a^2 + 2ab + b^2$
2) $(a-b)^2 = a^2 - 2ab + b^2$
3) $a^2 - b^2 = (a+b)(a-b)$
4) $a^2 + b^2 = (a+ib)(a-ib)$

eg. $(D^2+4) = (D+2i)(D-2i)$

5) $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
6) $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

eg. $(t-3)^3 = t^3 - 9t^2 + t\,27 - 27$

Trignometry

1) $\sin^2\theta + \cos^2\theta = 1$
2) $1 + \tan^2\theta = \sec^2\theta$
3) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$
4) $\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$
5) $\sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B$
6) $\cos(A+B) = \cos A \cos B - \sin A \cdot \sin B$
7) $\cos(A-B) = \cos A \cdot \cos B + \sin A \cdot \sin B$
8) $1 + \cos 2\theta = 2\cos^2\theta$

$1 - \cos 2\theta = 2\sin^2\theta$

9) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta \quad \Rightarrow 4\sin^3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow 4\cos^3\theta = \cos 3\theta + 3\cos\theta$

10) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

The Laplace transform of the function $f(t)$ for $t>0$ is denoted by $L\{F(t)\}$ and is given by

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

where s is Parameter

Note :- Notation

$$L\{F(t)\} = f(s) = F(s)$$

eg. $L\{y(t)\} = \frac{1}{s} = \bar{y}(s)$

Standard formulae of Laplace transform (LT)

1) $L\{K\} = \frac{K}{s}$; is K is constant

2) eg. $L\{1\} = \frac{1}{s}$, $L\{\frac{3}{2}\} = \frac{3}{2}s$

$$L\{\sqrt{5}\} = \frac{\sqrt{5}}{s}$$

2) $L\{t^n\} = \frac{n!}{s^{n+1}}$ where ① $n>0$

② $n! = 1\times2\times3\times4\times\cdots\times n!$

③ n is +ve integer

eg. $L\{t\} = \frac{1}{s^2}$

$$L\{t^5\} = \frac{5!}{s^6}$$

3) $L\{t^n\} = \frac{\overline{|n+1}}{s^{n+1}}$

where $\overline{|n+1} = \text{Gamma}(n+1)$

(ii) $\Gamma{n+1} = n!$, if n is +ve integer

(iii) $\Gamma{n+1} = n\,\Gamma{n}$

(iv) $\Gamma{\frac{1}{2}} = \sqrt{\pi}$

eg. $L\{t^{3/2}\} = \frac{\Gamma{\frac{3}{2}+1}}{s^{3/2+1}} = \frac{\frac{3}{2}\Gamma{\frac{3}{2}}}{s^{5/2}} = \frac{\frac{3}{2}\Gamma{\frac{1}{2}+1}}{s^{5/2}}$

$= \frac{\frac{3}{2}\times\frac{1}{2}\Gamma{\frac{1}{2}}}{s^{5/2}}$

$= \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{5/2}} = \frac{\frac{3}{4}\sqrt{\pi}}{s^{5/2}}$

$= \frac{3\sqrt{\pi}}{4s^{5/2}}$

eg: $L\{t^{7/2}\} = \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma{\frac{1}{2}}}{s^{9/2}}$

$= \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{9/2}}$

2) $L\{\sin at\} = \frac{a}{s^2+a^2}$

eg: $L\{\sin 5t\} = \frac{5}{s^2+25}$

$L\{\sin t\} = \frac{1}{s^2+1}$

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$\rightarrow$ $5L\{e^{-6t}\} + 7L\{\cos 2t\} + 3L\{\sinh 4t\}$

$5 \times \left(\frac{1}{s+6}\right) + 7 \times \left(\frac{s}{s^2+4}\right) + 3 \times \left(\frac{4}{7s^2+16}\right)$

$\frac{5}{s+6} + \frac{37s}{s^2+4} + \frac{12}{s^2-16}$

5) $L\{(3t-2)^2\}$

$\rightarrow$ $L\{(9t^2-12t+4)\}$

$9L\{t^2\} - 12L\{t\} + L\{4\}$

$9\left(\frac{2!}{s^3}\right) - 12\left(\frac{1}{s^2}\right) + \frac{4}{s}$

$\frac{18}{s^3} - \frac{12}{s^2} + \frac{4}{s}$

6) $L\{(5+4t)^3\}$

$\rightarrow$ $L\{125 + 300t + 240t^2 + 64t^3\}$

$\frac{125}{s} + \frac{300}{s^2} + \frac{240}{s^3} + \frac{64 \times 3!}{s^4}$

$\frac{125}{s} + \frac{300}{s^2} + \frac{240}{s^3} + \frac{384}{s^4}$

7) $L\{\sin(4-6t)\}$

$\rightarrow$ $L\{\sin(4-6t\}$

$L\{\sin 4 \cos 6t - \cos 4 \sin 6t\}$

$\sin 4\, L\{\cos 6t\} - \cos 4 L\{\sin 6t\}$

$\sin 4\left(\frac{s}{s^2+36}\right) - \cos 4\left(\frac{6}{36}\right)$

$= \frac{s \sin 4 - 6\cos 4}{s^2+36}$

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8) $L\{\cos(7t-2\}$

→ $L\{\cos 7t \cos 2 + \sin 7t \sin 2\}$

$\cos 2 \, L\{\cos 7t\} + \sin 2 \, \{\sin 7t\}$

$\cos 2 \left(\frac{s}{s^2+49}\right) + \sin 2 \left(\frac{7}{s^2+49}\right)$

$= \frac{s\cos 2 + 7\sin 2}{s^2+49}$

9) $L\{3\sin^2 t\}$

→ $3L\{\sin^2 t\}$

$3L\left\{\frac{1-\cos 2t}{2}\right\}$

$\frac{3}{2}L\{1-\cos 2t\} = \cancel{\frac{3}{2}L[\{1\}}$

$\frac{3}{2}\left(\frac{1}{s}\right) - \frac{3}{2}\left(\frac{s}{s^2+4}\right)$

$\frac{3}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$

10) $L\{4\cos^2 t\}$

$4L\{\cos^2 t\}$

$\frac{4}{2}L\{1+\cos 2t\}$

$2\left[\frac{1}{s} + \frac{1}{s^2+4}\right]$

) $L\{e^{-3t}\sin^2 t\}$

By 1st Shifting Property

$L\{e^{at} f(t)\} = [F(s)]_{s=s-a}$

Here . $F(t) = \sin^2 t$

$$F(s) = L\{F(t)\} = L\{\sin^2 t\}$$

$$= \frac{1}{2} L\{1 - \cos 2t\}$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$L\{e^{at} F(t)\} = \left[\frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)\right]$$

$$= \left[\frac{1}{2}\left(\frac{1}{s+3}\right) - \frac{s+3}{(s+3)^2+4}\right]$$

12) $L\{e^{5t} \cos^2 t\}$

By 1st Ch. Shifting Property

$$L\{e^{at} F(t)\} = [f(s)]_{s=s-a}$$

Here $F(t) = \cos^2 t$

$$L\{F(t)\} = L\{\cos^2 t\}$$

$$= \frac{1}{2} L\{1 + \cos 2t\}$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right]$$

$$L\{e^{at} F(t)\} = [F(s)]_{s=s-a=s-1}$$

$$= \frac{1}{2}\left[\frac{1}{s-1} + \frac{(s-1)}{(s-1)^2+4}\right]$$

Imp (All)

13) Find $L\{5\sin^3 t\}$

→ we know that

$\sin 3t = 3\sin t - 4\sin^3 t$

$\therefore 4\sin 3t = 3\sin t - 4\sin^3 t$

$\sin^3 t = \frac{1}{4}\left[3\sin t - \sin^3 t\right]$

$\therefore L\{5\sin^3 t\} = 5\,L\{\sin^3 t\}$

$= \frac{5}{4}\{3\sin t - \sin 3t\}$

$\therefore L\{5\sin^3 t\} = \frac{5}{4}\left[3\left(\frac{1}{s^2+1}\right) - \left(\frac{3}{s^2+9}\right)\right]$

$= \frac{\cancel{15}\ 15}{4}\left[\left(\frac{1}{s^2+1}\right) - \left(\frac{1}{s^2+9}\right)\right]$

14) ~~L{~~ $L\{2\cos^3 t\}$

$\cos 3t = 4\cos^3 t - 3\cos t$

$\therefore \cos^3 t = \frac{\cos 3t + 3\cos t}{4}$

$\cos^3 t = \frac{1}{4}\left[\cos 3t + 3\cos t\right]$

$\therefore L\{2\cos^3 t\} = 2L\{\cos^3 t\}$

$\therefore 2L\{\cos^3 t\} = \frac{2L}{4}\{\cos 3t + 3\cos t\}$

$\therefore L\{2\cos^3 t\} = \frac{2}{4}\left[\frac{s}{s^2+9} + 3\left(\frac{s}{s^2+1}\right)\right]$

$= \frac{2}{4}\left[\frac{s}{s^2+9} + \frac{\cancel{3s}\ 3s}{s^2+1}\right]$

$= \frac{2\cancel{s}s}{4}\left[\frac{1}{s^2+9} + \frac{3}{s^2+1}\right]$

* Laplace Transform by Multiplication Proce of t.

If $L\{F(t)\} = F(s)$

Then i) $L\{t\,F(t)\} = (-1)^1 \dfrac{d}{ds} F(s)$

ii) $L\{t^2 F(t)\} = (-1)^2 \dfrac{d^2}{ds^2} F(s)$

Note: i) $\dfrac{d}{dx}\dfrac{1}{x} = \dfrac{-1}{x^2}$

ii) $\dfrac{d}{dx}\dfrac{u}{v} = \dfrac{v\,du - u\,dv}{v^2}$

iii) $\dfrac{d}{dx}\, u.v = \cancel{v\,dv + u\,dv}\; u\,dv + v\,du$

15) ~~eg~~. $L\{t \cosh 4t\}$

Here, $F(t) = \cosh 4t$

$L\{F(t)\} = L\{\cosh 4t\}$

$F(s) = \dfrac{s}{(s^2 - a^2)} = \dfrac{s}{s^2 - 16}$

Using Property

$L\{t\,F(t)\} = (-1)\dfrac{d}{ds} F(s)$

$= (-1)\dfrac{d}{ds}\left(\dfrac{s}{s^2-16}\right)$

$= -\left[\dfrac{(s^2-16) - s\dfrac{d}{ds}(s^2-16)}{(s^2-16)^2}\right]$

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$$= -\left[\frac{(s^2-16)-s(2s)}{(s^2-16)^2}\right]$$

$$= -\left[\frac{(s^2-16)-2s^2}{(s^2-16)^2}\right]$$

$$= -\left[\frac{s^2-16-2s^2}{(s^2-16)^2}\right]$$

$$= -\left[\frac{-s^2-16}{(s^2-16)^2}\right]$$

$$= \frac{s^2+16}{(s^2-16)^2}$$

$$L\{t\cosh 4t\} = \frac{s^2+16}{(s^2-16)^2}$$

16) $L\{t^2 e^{5t}\}$

$$L\{t^2 F(s)\} = (-1)^2 \frac{d^2}{ds^2} F(s)$$

$$\cancel{L\{t^2}\ F(s) = e^{5t} = \frac{1}{s-5}$$

Using Property

$$L\{t^2 F(t)\} = (-1)^2 \frac{d^2 F(s)}{ds^2}$$

$$= \frac{d^2}{ds^2}\left[\frac{1}{s-5}\right]$$

$$= \frac{-1}{[s-5]^2}$$

17) $L\{t \sin 2t\}$

$$L\{t^2 F(t)\} = (-1)^2 \frac{d^2}{ds^2} F(s)$$

$\cancel{f(t)} = f(t) = \sin 2t$, $F(s) = \frac{2}{s^2+4}$

$$\therefore \{t \sin 2t\} = (-1)\frac{d}{ds}\frac{2}{s^2+4}$$

$$= \frac{2}{(s^2+4)^2}$$

18) $L\{\sqrt{t}\} = L\{t^{1/2}\} = \frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$ $\left(\Gamma n+1 = n\Gamma n\right.$

$$= \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}}$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}}$$

* Laplace Transform by division of Powers of t

If, $L\{f(t)\} = F(s)$

then $L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

Note :- i) $\int \frac{1}{s-a}\,ds = \log|s-a| + c$

ii) $\int \frac{1}{s^2+a^2}\,ds = \frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right) + c$

$$\int \frac{F'(x)}{F(x)} = \log F(x)$$

~~iii)~~ ~~Eg. [19]~~ ~~Find~~ $L\left\{ e^{-t}\, \frac{\sin^2 t}{t} \right\}$

Here. $F(t) = \sin^2 t$

$[1-\cos 2t = 2\sin^2 t]$

$$\therefore F(t) = \frac{1}{2}[2\sin^2 t]$$

$$= \frac{1}{2}[1-\cos 2t]$$

$$\therefore L\{F(t)\} = \frac{1}{2} L\{1-\cos 2t\}$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

Now.

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)\,ds$$

$$L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds$$

$$= \frac{1}{2}\int_s^\infty \frac{1}{s} - \frac{s}{s^2+4}\, ds$$

$$= \frac{1}{2}\Big[\log s - \log|s^2+4|\Big]_s^\infty$$

$$= \frac{1}{2}\Big[\log s - \log|s^2+4|\Big]$$

$\frac{1}{\infty} = 0$

multiply by t → derrivation
divide by t → Integration

If Derrivation → multiply by s
Integration → Divide by s

$\log 1 = 0$

$e^0 = 1$

$\log 1 - \log m = \log \frac{1}{m}$

$$= \frac{1}{2} \int_s^\infty \frac{1}{s} - \frac{1}{2} \cdot \frac{2s}{s^2+4} \, ds$$

$$= \frac{1}{2} \left[\log s - \frac{1}{2} \log |s^2+4| \right]_s^\infty$$

$$= \frac{1}{2} \left[\log s - \log (s^2+4)^{1/2} \right]_s^\infty$$

$$= \frac{1}{2} \left[\log s - \log \sqrt{s^2+4} \right]_s^\infty$$

$$= \frac{1}{2} \left[\frac{\log s}{\log \sqrt{s^2+4}} \right]_s^\infty$$

$$= \frac{1}{2} \left[\log \frac{s}{\sqrt{s^2+4}} \right]_s^\infty = \frac{1}{2} \left[\log \frac{s}{\sqrt{s^2} \sqrt{1+\frac{4}{s^2}}} \right]_s^\infty$$

$$= \frac{1}{2} \left[0 - \log \frac{1}{\sqrt{1+\frac{4}{s^2}}} \right]_s^\infty$$

$$= \frac{1}{2} \left[0 - \log \frac{1}{\sqrt{1+\frac{4}{s^2}}} \right] = \frac{1}{2} \left[0 - \log \frac{s}{\sqrt{s^2+4}} \right]$$

$$= \frac{1}{2} \left[0 - \log \frac{s}{\sqrt{s^2+4}} \right]$$

$$= \frac{1}{2} \log \frac{\sqrt{s^2+4}}{s}$$

Eg. 22) Find $L\left\{ \frac{\cos 3t - \cos 5t}{t} \right\}$

Eg. 23) Find $L\left\{ \frac{e^{7t} - e^{5t}}{t} \right\}$

eg. 23) ~~$L\{\cos 3t - \cos 5t$~~

$$L\left\{\frac{e^{7t} - e^{5t}}{t}\right\}$$

$$F(t) = e^{7t} - e^{5t}$$

$$L\{F(t)\} = L\{e^{7t} - e^{5t}\}$$

$$= \left[\frac{1}{s-7} - \frac{1}{s-5}\right]$$

$$\text{Now } L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)\, ds$$

$$L\left\{\frac{e^{7t} - e^{5t}}{t}\right\} = \int_s^\infty \frac{1}{s-7} - \frac{1}{s-5}\, ds$$

$$= \Big[\log(s-7) - \log(s-5)\Big]_s^\infty$$

$$= \left[\log\frac{s-7}{s-5}\right]_s^\infty$$

$$= \log\left[\frac{s-5}{s-7}\right]$$

22) Find $L\left\{\frac{\cos 3t - \cos 5t}{t}\right\}$

$$F(t) = \cos 3t - \cos 5t$$

$$L\{F(t)\} = L\{\cos 3t - \cos 5t\}$$

$$= \left[\frac{s}{s^2+9} - \frac{s}{s^2+25}\right]$$

Now,

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

$$L\left\{\frac{\cos 3t - \cos 5t}{t}\right\} = \int_s^\infty \frac{s}{s^2+9} - \frac{s}{s^2+25}\,ds$$

$$= \frac{1}{2}\Big[\log|s^2+9| - \log|s^2+25|\Big]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2+9}{s^2+25}\right]$$

$$= \frac{1}{2}\left[\log\frac{s^2+25}{s^2+9}\right]$$

ex: 24) $L\left\{\frac{\cos 5t - \cos 3t}{t}\right\}$

$$F(t) = \cos 5t - \cos 3t$$

$$L\{F(t)\} = L\{\cos 5t - \cos 3t\}$$

$$= \frac{s}{s^2+25} - \frac{s}{s^2+9} \Rightarrow F(s)$$

Now

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

$$L\left\{\frac{\cos 5t - \cos 3t}{t}\right\} = \int_s^\infty \frac{s}{s^2+25} - \frac{s}{s^2+9}\,ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{s^2+25} - \frac{2s}{s^2+9}\, ds$$

$$= \frac{1}{2}\left[\log|s^2+25| - \log|s^2+9|\right]_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{s^2+25}{s^2+9}\right]$$

$$= \frac{1}{2}\left[\log\left\{\frac{s^2+9}{s^2+25}\right]\right.$$

$$= \log\sqrt{\frac{s^2+9}{s^2+25}}$$

Laplace transform of derrivative:

If $L\{y(t)\} = Y(s)$, then

i) $L\{y'\} = L\left\{\frac{dy}{dt}\right\} = sY(s) - y(0)$

~~ii) $L\{y''\} =$~~

ii) $L\{y''\} = L\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0)$

iii) $L\{y'''\} = L\left\{\frac{d^3y}{dt^3}\right\} = \cancel{s^3 Y(s) - s^2y(0) - s(y)(0)}$

$= s^3Y(s) - s^2y(0) - s\{y'(0)\} - y''(0)$

Laplace Transform of Integration

If $L\{y(t)\} = Y(s)$ then

i) $L\left\{\int_0^t f(u)\,du\right\} = \dfrac{F(s)}{s}$

ii) $L\left\{\int_0^t\int_0^t F(u)\,du\right\} = \dfrac{F(s)}{s^2}$

Exam question W-2019

eg 25 Find $L\left\{\cosh t \int_0^t e^u \cosh u\,du\right\}$

eg. 26. Evaluate $\int_0^\infty e^{-at}\,\dfrac{\sin^2 t}{t}\,dt$

W-2022 eg. 27. Evaluate $\int_0^\infty \dfrac{\cos 4t - \cos 3t}{t}\,dt$

eg 28. Find $L\{e^{-3t}\sin^2 t\}$

eg 29. Find L.T of $F(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < 2 \end{cases}$